MATH 3235 Probability Theory

Review 9/29

Probability Space.

Ω , $\mathcal{F}$, $\mathbb{P}$ ← prob.

Ω: Sample space

$\mathcal{F}$: Space of events

if $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

if $A_i \in \mathcal{F} \Rightarrow \bigcup_i A_i \in \mathcal{F}$

$\mathbb{P} : \mathcal{F} \to \mathbb{R}$

$\mathbb{P}(\Omega) = 1$

$A_i \in \mathcal{F}$ and $A_i \cap A_j = \emptyset$ $i \neq j$

$\Downarrow$

$$\mathbb{P}\left(\bigcup_i A_i\right) = \sum_i \mathbb{P}(A_i)$$

$$\mathbb{P}(A \mid B) = \mathbb{P}(A \cap B) / \mathbb{P}(B)$$

$\mathbb{P}(\cdot \mid B)$ is a prob

Partition Thm.

A_i are disjoint $\quad A_i \cap A_j = \phi \quad i \neq j$

$$\bigcup_i A_i = \Omega$$

$$\mathbb{P}(B) = \sum_i \mathbb{P}(B \mid A_i) \mathbb{P}(A_i)$$

Bayes Thm

$$\mathbb{P}(A_i \mid B) = \frac{\mathbb{P}(B \mid A_i) \mathbb{P}(A_i)}{\sum_j \mathbb{P}(B \mid A_j) \mathbb{P}(A_j)}$$

$A \perp\!\!\!\perp B \qquad \mathbb{P}(A \cap B) = \mathbb{P}(A) \mathbb{P}(B)$

$$A \perp\!\!\!\perp B \implies A^c \perp\!\!\!\perp B$$

$$\mathbb{P}(A^c \cap B) + \mathbb{P}(A \cap B) = \mathbb{P}(B)$$

$$\mathbb{P}(A^c \cap B) = -\mathbb{P}(A)\mathbb{P}(B) + \mathbb{P}(B) =$$

$$= \mathbb{P}(B)(1 - \mathbb{P}(A)) =$$

$$= \mathbb{P}(B)\,\mathbb{P}(A^c)$$

$$A \perp\!\!\!\perp B^c \qquad A^c \perp\!\!\!\perp B^c$$

X_1 X_2 are Bernoulli r.v.

$$\operatorname{cov}(X_1, X_2) = 0 \implies X_1 \perp\!\!\!\perp X_2$$

$$A = \{X_1 = 1\} \qquad B = \{X_2 = 1\}$$

$$\mathbb{E}(X_1 X_2) = \mathbb{P}(A \cap B)$$

$$\mathbb{E}(X_1) = \mathbb{P}(A) \qquad \mathbb{E}(X_2) = \mathbb{P}(B)$$

$$\operatorname{cov}(X_1, X_2) = 0$$

$$\mathbb{E}(X_1 X_2) - \mathbb{E}(X_1)\mathbb{E}(X_2) = 0$$

$$\{X_1 = 1\} \perp\!\!\!\perp \{X_2 = 1\}$$

$$\{X_i = 0\} = \{X_i = 1\}^c$$

$$\{X_1 = 1\} \perp\!\!\!\perp \{X_2 = 0\}$$

Random variables.

$$X : \Omega \to \mathbb{R}$$

$$X^{-1}((a, b]) \in \mathcal{F}$$

Discrete r.v.

$\operatorname{Im}(X)$ is countable

$$X^{-1}(x) \in \mathcal{F} \qquad x \in \operatorname{Im}(X)$$

$$P_X(x) = \mathbb{P}(X = x)$$

Family of r.v.

Bernoulli $\quad P_X(0) = 1-p \quad P_X(1) = p$

Binonical $\quad$ sum of Bernoulli

$$P_X(x) = \binom{N}{x} p^x (1-p)^{N-x}$$

Geometric

$$P_X(x) = (1-p)^{x-1} p$$

Poisson

$$P_X(x) = \frac{\lambda^x}{x!} e^{-\lambda}$$

(limit of Binonial $N \to \infty$

$p = \frac{\lambda}{N}$)

$$\mathbb{E}(X) = \sum_x x\, p(x)$$

$$\mathbb{E}(g(X)) = \sum_x g(x)\, p(x)$$

$$\mathbb{P}(g(X) = y) = \sum_{x \mid g(x) = y} p(x)$$

$$\mathbb{E}(X \mid B) = \sum_x x\, \mathbb{P}(X = x \mid B)$$

$$\mathbb{E}(X) = \sum_i \mathbb{E}(X \mid B_i)\, \mathbb{P}(B_i)$$

if B_i is a partition.

Jointy distributed r.v.

X_i $\quad i = 1 \ldots N$

$$p(x_1, \ldots, x_N) = \mathbb{P}(X_1 = x_1, X_2 = x_2, \ldots X_N = x_N)$$

Marginals

$X \quad Y$

$$p_X(x) = \sum_y p(x, y)$$

$X \perp\!\!\!\perp Y$ independent

if $\quad p(x,y) = p_X(x)\, p_Y(y)$

$X \quad Y$ independent

$Z = X + Y$

$$p_Z(z) = \sum_x p_X(x)\, p_Y(z-x)$$

Convolution.

$f \quad g$

$$(f * g)(k) = \sum_{\substack{n,m \\ n+m=k}} f(n)\, g(m) =$$

$$= \sum_n f(n)\, g(k-n)$$

---

p.g.f

$$G_X(s) = \mathbb{E}(s^X)$$

$$G_{X+Y}(s) = G_X(s)\, G_Y(s)$$

if $X \perp\!\!\!\perp Y$

$$p(x) = C x^{\alpha} \qquad x \geq 1$$

$$\alpha < -1 \qquad \sum_{x \geq 1} x^{\alpha} < +\infty$$

$$C = \left(\sum_{x \geq 1} x^{\alpha} \right)^{-1}$$

$\mathbb{E}(X)$ finite

$$\mathbb{E}(X) = C \sum_{x \geq 1} x^{\alpha+1}$$

$$\alpha + 1 < -1 \qquad \alpha < -2$$

$$x \in \mathbb{Z}$$

$$p(x) = \frac{C}{(|x| + 1)^{\alpha}}$$

$$\sum \frac{1}{(|x| + 1)^{\alpha}} < +\infty$$

$$\alpha > 1$$

$$\mathbb{E}(X) = C \sum_x \frac{x}{(|x|+1)^{\alpha}}$$

$\alpha = 3$

$$= C \sum_x \frac{x}{(|x|+1)^3}$$

$$\frac{x}{(|x|+1)^3} \simeq \frac{1}{x^2} \quad \text{for } x \text{ large}$$

$\alpha = 3/2$

$$\mathbb{E}(X) = \sum_x \frac{x}{(1+|x|)^{3/2}}$$

$$\lim_{N,M} \sum_{x=-M}^{N} \frac{x}{(1+|x|)^{3/2}}$$

0

$$Z = \sum_{i=1}^{M} X_i$$

X_i are i.i.d.

ind from M

$$G_Z(s) = G_M\left(G_X(s)\right)$$

M a r.v. That Take value m

with prob 1

$$G_M(s) = s^m$$

$$Z = \sum_{i=1}^{M} X_i = G_M(G_X(s)) =$$

$$= \left(G_X(s)\right)^m$$
